It is not Possible to Perturb the N-wrapped Critical Catenoid by a Family of Immersed Free Boundary Minimal Surfaces in the Unit Ball

Luca Seemungal

Supervisor: Lucas Ambrozio

Warwick Mathematics Institute

Undergraduate Research Support Scheme

Introduction: free boundary minimal surfaces, and a PDE problem

Minimal surfaces are critical points of the *area functional*; i.e. a surface X is *minimal* if given any one-parameter family of variations X^t with $X^0 = X$ (and possibly some boundary conditions), we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Area}[X^t] = 0.$$

In other words, an infinitesimally slight perturbation of the surface should not change the surface area. For example, if we wish to find a minimal surface with fixed boundary, then our variations X^t would have to share that same boundary. If, however, part or all of the boundary is permitted to move across a surface, then this same boundary condition is also imposed on the variations X^t; such a surface is said to have *free boundary*.

We are in particular interested in the *critical catenoid* (Fig. 1), which is a minimal surface that has free boundary in the surface of the unit sphere. We have the conformal chart

$$X_N : [-L, L] \times [-N\pi, N\pi] \ni (t, \theta) \mapsto (a\cosh t\cos\theta, a\cosh t\sin\theta, at) \in \mathbb{B}^3$$

where $N \in \mathbb{N}$ is the number of times we wrap around the critical catenoid, a is the unique positive solution to the equation

$$\mathrm{arccosh}^2(1/x) = 1/(1-x^2),$$

and L is the unique positive solution to the equation

$$x = \coth x.$$

The question as to whether there are other free boundary minimal annuli (surfaces that "look like a cylinder") in the unit ball is open.

Let us call "the critical catenoid wrapped around itself N times" "the N-wrapped critical catenoid". Our project tried to answer the question:

Can we vary the N-wrapped critical catenoid by a family of minimal annuli with free boundary in the unit sphere?

Clearly, we *can* - just rotate the critical catenoid in the x and y axes. We call these *trivial solutions*. But can we do this non-trivially? If we can, then the following PDE problem for the *normal speed of the variations* ϕ - the speed at which the variation moves perpendicularly to the critical catenoid - must have non-trivial solutions (where in this case, "non-trivial" does *not* mean "$\phi = 0$", but rather it means "the ϕ that corresponds to a rotation of the critical catenoid"!):

$$\begin{aligned} \left(\Delta + \frac{2}{\cosh^2(t)}\right)\phi = 0 \qquad &\text{for all } t \in (-L, L) \\ \phi(-L,\theta) = -L\partial_t\phi(-L,\theta)& \\ \phi(L,\theta) = L\partial_t\phi(L,\theta).& \end{aligned} \tag{1}$$

Methods 1: Separation of Variables and a Sturm-Liouville Problem

Using separation of variables yields, among other things, the following well-studied Sturm-Liouville problem:

$$\begin{aligned} T'' + \frac{2}{\cosh^2(t)}T &= \lambda T \\ T'(-L)/T(-L) &= -1/L \\ T'(L)/T(L) &= 1/L, \end{aligned} \tag{2}$$

where λ are the eigenvalues for the S-L problem. This technique also gives the condition that λ is the square of a rational and so $\lambda \geq 0$. The case $\lambda = 0$ has already been explored in [MNS13], and in Subsection 6.3 they showed that $\lambda = 0$ yeilds no solutions.

Now, suppose $\gamma(t)$ is a solution to the ODE problem (2). Then we can decompose $\gamma(t)$ into an odd solution and an even solution; therefore we search for odd and even solutions; the following two solutions to the ODE are well known, which do not yet satisfy the boundary conditions:

$$\begin{aligned} O_\lambda(t) &= -\lambda^{1/2}\sinh(\lambda^{1/2}t) + \cosh(\lambda^{1/2}t)\tanh(t) \\ E_\lambda(t) &= \lambda^{1/2}\cosh(\lambda^{1/2}t) - \sinh(\lambda^{1/2}t)\tanh(t), \end{aligned} \tag{3}$$

We notice that they are linearly independent and so form a basis for the solution space of the ODE in (2). However, we have yet to establish whether or not they satisfy the boundary conditions.

We also note that mode $\lambda = 1$ corresponds to the trivial solutions - rotations of the critical catenoid.

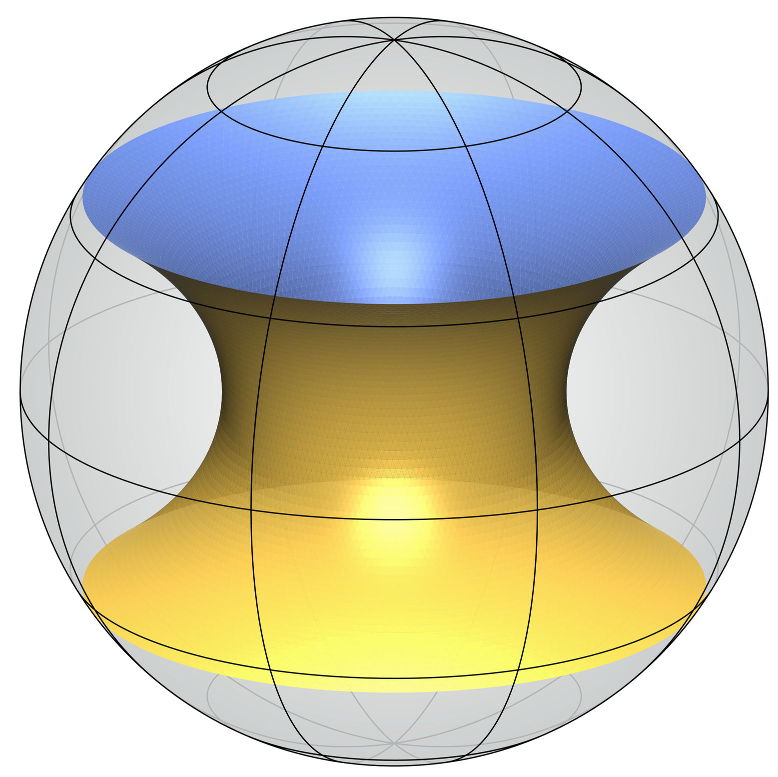

Fig. 1: Image by Mario B. Schulz, `https://mbschulz.github.io`

Methods 2: Shooting Method

Now we wish to see if these solutions have the required boundary condition. We use the *shooting method*: we "shoot" the solution from a known initial point and see if it hits the boundary condition.

For the odd case, the shooting method boils down to finding the zeroes of

$$f_O(\lambda) := \frac{O'_\lambda(L)}{O_\lambda(L)} - \frac{1}{L}.$$

Fig. (2) suggests that it has two roots. Calculating this explicitly, we deduce the following equation for $\lambda^{1/2}$:

$$\lambda^{1/2} = \left(\tanh(\lambda^{1/2}L) \pm \sqrt{\tanh^2(\lambda^{1/2}L) + L^2 - 2}\right) \Big/ L. \tag{4}$$

For the even case, we have

$$f_E(\lambda) := \frac{E'_\lambda(L)}{E_\lambda(L)} - \frac{1}{L}.$$

We prove that if f_E has a root, then it cannot be rational. Calculating $f_E(\lambda) = 0$ explicitly we obtain

$$\lambda^{1/2} = \left(\coth(\lambda^{1/2}L) \pm \sqrt{\coth^2(\lambda^{1/2}L) + L^2 - 2}\right) \Big/ L. \tag{5}$$

We deduce from these equations that in each case, λ must be irrational.

Transcendental Number Theory

If $a \in \mathbb{C}$ is algebraic, then the *Lindemann-Weierstrass theorem* tells us that e^a must be transcendental. Furthermore since the algebraic numbers are an algebraically closed field, any algebraic combination of a transcendental number must also be transcendental.

Firstly, we can deduce that L must be transcendental: if L is algebraic, then e^L is transcendental. But $\coth(L)$ is an algebraic combination of e^L, and so $\coth(L) = L$ is transcendental, which is a contradiction. So L is transcendental. We can furthermore deduce that e^L is transcendental: if e^L were algebraic, then $L = \coth(L)$ would also be algebraic, a contradiction.

Now, if $\lambda^{1/2}$ is *rational* then we notice that the right-hand side of both Eqns. (4) and (5) are algebraic combinations of e^L. Therefore, since e^L is transcendental, the right-hand sides are transcendental, and so in both cases $\lambda^{1/2}$ must be transcendental. This is a contradiction to the rationality of $\lambda^{1/2}$, and so $\lambda^{1/2}$ must be irrational.

Conclusions

Our analysis shows that the only solutions to the PDE problem (1) are those given by rotations of the critical catenoid. Therefore it is not possible to perturb any N-wrapped critical catenoid by a family of immersed free boundary minimal annuli.

These results strongly suggest that if we think of the set of immersed free boundary minimal annuli in the unit ball as an abstract space, then the N-wrapped critical catenoids are *isolated* points.

References

[MNS13] Davi Maximo, Ivaldo Nunes, and Graham Smith. *Free boundary minimal annuli in convex three-manifolds*. 2013. arXiv: `1312.5392 [math.DG]`.

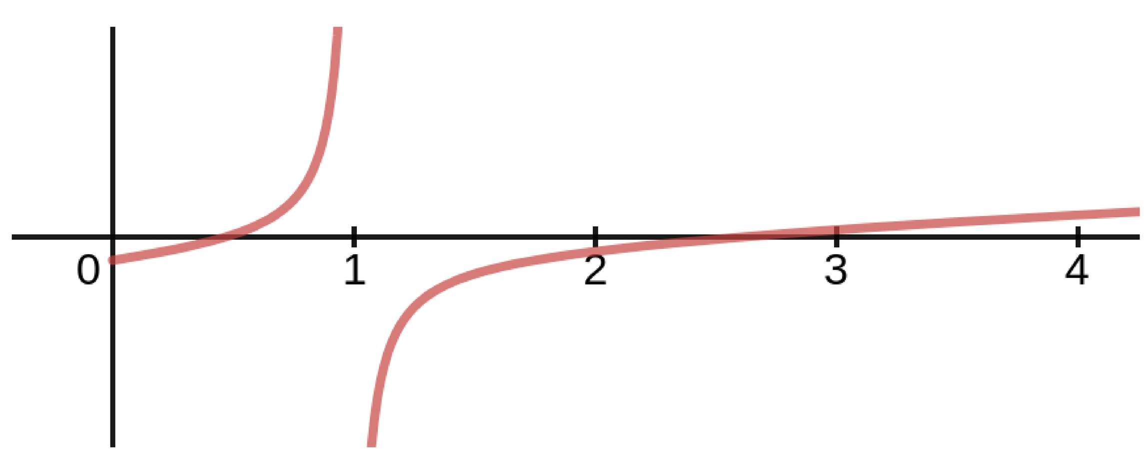

Fig. 2: Plot of $f_O(\lambda)$ with $\lambda^{1/2}$ on the horizontal axis. Its roots are irrational. Made with the wonderful tool Desmos.

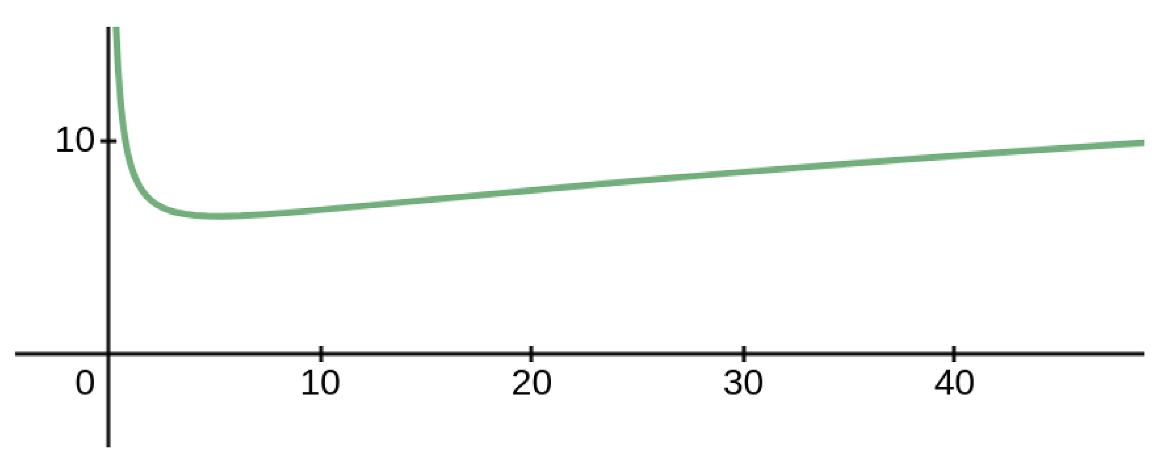

Fig. 3: Plot of $f_E(\lambda)$ with $\lambda^{1/2}$ on the horizontal axis. It doesn't have any roots. Made with the wonderful tool Desmos.